SOME REMARKS ON CONFORMAL REPRESENTATION.*

By T. H. Gronwall.

1. In recent investigations of the uniformization of analytic functions, the following proposition† is of fundamental importance:

When the analytic function

$$z = f(x) = \sum_{n=1}^{\infty} a_n x^n \tag{1}$$

effects the conformal representation of the circle $|x| < 1$ *on a simple region*‡ *in the z-plane, the area of this region being less than or equal to A, then, for every* $r < 1$, *the expressions* $|f(x)|$ *and* $|f'(x)|$ *have upper boundaries for* $|x| \leqq r$ *which depend on r and A only, and not on the coefficients in* (1).

Koebe gives the inequality

$$|f(x)| < \sqrt{A}\left(\frac{r}{\pi(1-r)^2} + 1\right) \qquad \text{for} \qquad |x| \leqq r < 1$$

and Courant the corresponding inequality for $f'(x)$

$$|f'(x)| \leqq \sqrt{\frac{A}{\pi}} \cdot \frac{1}{1-r^2}.$$

We shall now show that *for* $|x| \leqq r < 1$

$$\begin{aligned} |f(x)| &\leqq \sqrt{\frac{A}{\pi}} \cdot \sqrt{\log \frac{1}{1-r^2}}, \\ |f'(x)| &\leqq \sqrt{\frac{A}{\pi}} \cdot \frac{1}{1-r^2}, \end{aligned} \tag{2}$$

and that these upper boundaries cannot be replaced by any smaller ones.

Under the assumption made on $f(x)$, the circle $|x| < r$ is conformally represented on a simple region in the z-plane, the area of which is

$$A(r) = \int_0^r d\rho \int_0^{2\pi} |f'(\rho e^{\theta i})|^2 \rho \, d\vartheta = \pi \sum_{n=1}^{\infty} n \, |a_n|^2 r^{2n}, \tag{3}$$

and since this region is interior to that corresponding to the circle $|x| < 1$,

* Read before the American Mathematical Society, October 31, 1914.

† Koebe, Journal für Mathematik, 138 (1910), p. 222. Courant, Math. Annalen, 71 (1912), p. 164.

‡ Simple region = schlichter Bereich, a simply connected and nowhere overlapping region.

we have

$$\pi \sum_{n=1}^{\infty} n \mid a_n \mid^2 r^{2n} < A$$

for any $r < 1$. A fortiori, the inequality

$$\pi \sum_{n=1}^{N} n \mid a_n \mid^2 r^{2n} < A,$$

subsists, and letting r tend toward unity, we obtain

$$\pi \sum_{n=1}^{N} n \mid a_n \mid^2 \leqq A,$$

whence, observing that the terms of the series are positive, we find by increasing N indefinitely

$$\pi \sum_{n=1}^{\infty} n \mid a_n \mid^2 \leqq A. \tag{4}$$

On the other hand, it is evident that for $\mid x \mid \leqq r$

$$\mid f(x) \mid \leqq \sum_{n=1}^{\infty} \mid a_n \mid r^n. \tag{5}$$

In the Lagrange inequality

$$\left(\sum_{n=1}^{\infty} u_n v_n \right)^2 \leqq \sum_{n=1}^{\infty} u_n^2 \cdot \sum_{n=1}^{\infty} v_n^2,$$

where u_n, v_n are any positive quantities such that the series are convergent, we now make $u_n = \sqrt{n} \mid a_n \mid$, $v_n = r^n / \sqrt{n}$ and obtain from (5),

$$\mid f(x) \mid^2 \leqq \sum_{n=1}^{\infty} n \mid a_n \mid^2 \cdot \sum_{n=1}^{\infty} \frac{r^{2n}}{n} = \sum_{n=1}^{\infty} n \mid a_n \mid^2 \cdot \log \frac{1}{1 - r^2},$$

whence, using (4), we immediately obtain the first inequality (2). To show that the upper boundary obtained is the smallest possible, consider the special function

$$f_0(x) = \sqrt{\frac{A}{\pi}} \cdot \frac{\log \dfrac{1}{1 - r_0 x}}{\sqrt{\log \dfrac{1}{1 - r_0^2}}},$$

where $0 < r_0 < 1$. This function effects the conformal representation of the unit circle on a simple region, since for any two different values of x inside the unit circle the values of $f_0(x)$ are different, and from (3) it is seen at once that the area of this region equals A. Furthermore, for $x = r_0$, we have

$$\mid f_0(r_0) \mid = \sqrt{\frac{A}{\pi}} \cdot \sqrt{\log \frac{1}{1 - r_0^2}},$$

so that the upper boundary (2) is actually reached for the particular value $r = r_0$.

To prove the second inequality (2), we observe that for $|x| \leqq r$

$$|f'(x)| \leqq \sum_{n=1}^{\infty} n\, |a_n|\, r^{n-1};$$

applying Lagrange's inequality with $u_n = \sqrt{n}\,|a_n|$, $v_n = \sqrt{n}\, r^{n-1}$, we find

$$|f'(x)|^2 \leqq \sum_{n=1}^{\infty} n\, |a_n|^2 \cdot \sum_{n=1}^{\infty} n r^{2n-2} = \sum_{n=1}^{\infty} n\, |a_n|^2 \cdot \frac{1}{(1-r^2)^2},$$

and combining with (4), the desired result is obtained. Using the comparison function

$$f_0(x) = (1 - r_0^2)\sqrt{\frac{A}{\pi}} \cdot \frac{x}{1 - r_0 x}$$

it is seen as before that the upper boundary cannot be replaced by any smaller one.

2. In connection with his exposition of Koebe's distortion theorem, Fricke* proves the following proposition:

When

(6) $$f(x) = \frac{1}{x} + \sum_{n=1}^{\infty} a_n x^n,$$

where the power series has no constant term and converges for $|x| < 1$, *and* $f(x)$ *does not take the same value for two different values of* x *inside the unit circle, then the maximum* $M(r)$ *of* $|f(x)|$ *for* $|x| = r$ *satisfies the inequality*

$$M(r) < \frac{2 + 2\sqrt{2}}{r}, \qquad 0 < r < 1.$$

We shall now show that *this inequality may be replaced by*

(7) $$M(r) < \frac{9}{4} \cdot \frac{1}{r}.$$

Since $f(x)$ takes each of its values but once in the unit circle, $z = f(x)$ effects the conformal representation of the unit circle on a simple region in the z-plane, containing the point $z = \infty$ (corresponding to $x = 0$) in its interior; to the circumference $|x| = r$ $(0 < r < 1)$ there corresponds a closed curve C_r, all the points of which are at finite distance, and if $r > r_0$, the curve C_r lies entirely inside the curve C_{r_0}. Consequently

* R. Fricke and F. Klein, Vorlesungen über die Theorie der automorphen Functionen, vol. 2 (Leipzig, Teubner, 1912), pp. 497–498.

$M(r) < M(r_0)$, and

$$rM(r) < rM(r_0) < M(r_0), \qquad 0 < r_0 < r < 1.$$

But since $xf(x)$ is holomorphic for $|x| < 1$, we have for $r \leqq r_0$

$$rM(r) = \max_{|x|=r} |xf(x)| \leqq \max_{|x|=r_0} |xf(x)| = r_0 M(r_0) < M(r_0),$$

and this, together with the preceding inequality, gives

$$rM(r) < M(r_0), \qquad 0 \leqq r < 1. \tag{8}$$

We shall now establish an upper boundary for $M(r_0)$, and then determine r_0 so as to make this boundary as small as possible. The area inside the curve C_r is readily found to be

$$\pi\left(1 - \sum_{n=1}^{\infty} n\,|a_n|^2 r^{2n}\right),$$

and this area being obviously positive for every $r < 1$, we find, letting r tend toward unity in the same way as when proving (4), that

$$\sum_{n=1}^{\infty} n\,|a_n|^2 \leqq 1. \tag{9}$$

Now

$$M(r) \leqq \frac{1}{r} + \sum_{n=1}^{\infty} |a_n|\, r^n,$$

and applying the Lagrange inequality and the relation (9) to the right hand series in the same way as before, we see that

$$M(r) \leqq \frac{1}{r} + \sqrt{\log \frac{1}{1 - r^2}}\,.$$

In the interval $0 \leqq r \leqq 1$, the last expression has only one minimum $= 2.242 \cdots$ for $r = 0.74 \cdots$; taking the latter value for r_0, it follows from (8) that

$$rM(r) < 2.242 \cdots < \frac{9}{4} \qquad \text{for} \qquad 0 < r < 1,$$

which proves our statement (6).

The particular function

$$f(x) = \frac{1}{x} + e^{2\alpha i} x, \tag{10}$$

where α is real, effects a conformal representation of the unit circle on that part of the z-plane which lies outside of the rectilinear cut from

$z = -2e^{ai}$ to $z = 2e^{ai}$. For this function we have

$$M(r) = \frac{1}{r} + r \leqq \frac{2}{r} \qquad \text{for} \qquad 0 < r \leqq 1,$$

so that the constant in (7) cannot be replaced by any value less than 2. It is probable that, for every function (6) satisfying the conditions stated, $M(r) \leqq \frac{1}{r} + r$, the equality sign holding only when $f(x)$ has the form (10). The proof of this statement seems very difficult; the case, however, when all a_n are real and not negative, may be disposed of as follows. From the assumptions concerning $f(x)$, it is seen that $f'(x) \neq 0$ inside the unit circle, and consequently

$$f'(r) = -\frac{1}{r^2} + \sum_{n=1}^{\infty} n a_n r^{n-1} < 0 \qquad \text{for} \qquad 0 < r < 1,$$

since otherwise $f'(x) = 0$ would have a positive root $\leqq r$. Consequently, letting r tend toward unity in the same way as before, we find

$$\sum_{n=1}^{\infty} n a_n \leqq 1,$$

or

$$a_1 \leqq 1 - \sum_{n=2}^{\infty} n a_n,$$

whence

$$M(r) = \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leqq \frac{1}{r} + r - \sum_{n=2}^{\infty} a_n(nr - r^n) \leqq \frac{1}{r} + r,$$

the equality sign holding only for $a_1 = 1$ and $a_n = 0$, $(n > 1)$.

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